CALIFORNIA STATE UNIVERSITY, BAKERSFIELD
Lee Webb Math Field Day 2013
Team Medley, Junior-Senior Level

Each correct answer is worth ten points. Answers require justification. Partial credit may be given. Unanswered questions are given zero points.

You have 50 minutes to complete the Exam. When the exam is over, give only one set of answers per team to the proctor. Multiple solutions to the same problem will invalidate each other.

Elegance of solutions may affect score and may be used to break ties.

All calculators, cell phones, music players, and other electronic devices should be put away in backpacks, purses, pockets, etc. Leaving early or otherwise disrupting other contestants may be cause for disqualification.

1. A coin is loaded so that it flips Heads 45% of the time. However, this is the only coin that anybody can find to determine which team kicks off in the Slide Rule Bowl. Nonetheless, the referee explains a procedure for using the coin that easily satisfies both team captains that they each have a 50% chance of winning the toss. Describe such a procedure.

Solution: The referee's plan is to define one turn as consisting of two flips. If the two flips are both heads or both tails, the result is thrown out. Of the two remaining possibilities - Heads then Tails or Tails then Heads – each has exactly the same probability, so assign one of these as Team A winning and the other as Team B winning.

2. Alicia, Brian, Charlie, Darlene, Erica, Frank, and Gabrielle are all either engineers or sales representatives. It is well-known that the engineers always lie and the sales representative always tell the truth. Brian and Erica are sales representatives. Charlie says that Darlene is an engineer. Alicia says that Brian says that Charlie says that Darlene says that Erica says that Frank says that Gabrielle is not a sales representative. If Alicia is an engineer, how many sales representatives are there?

Solution: There are seven people. A priori, there are 128 possibilities for how the jobs sort out. But 3 people's jobs are specified, so we are already down to 16 possibilities for the rest of them. "Charlie says that Darlene is an engineer" implies that Charlie and Darlene have different jobs – and we are down to only 8 possibilities for the job distribution. "Frank says that Gabrielle is not a sales representative" is true when Frank and Gabrielle have different jobs and false if they have the same jobs. Working back up a chain of "says" the truth value is only changed when it is an engineer saying it. Before Frank there are, in all remaining cases, two engineers. So, we need Frank and Gabrielle to have different jobs. This gets us down to four cases left. All our of these have three engineers and four sales representatives.

3. Suppose $0 < x_1 \le x_2 \le x_3 \le ... \le x_n$ and $0 < y_1 \le y_2 \le y_3 \le ... \le y_n$. Show that
$$\left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n y_i\right) \le n \sum_{i=1}^n x_i y_i \ .$$

Solution: Start with the observation that if $0 < a \le b$ and $0 \le c < d$ then $ad + bc \le ac + bd$ since this inequality is equivalent to $ac + bd - ad - bc = (b-a)(d-c)$ and the last is the product of two

non-negative numbers. Using this fact, gives

$$\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i y_i + \sum_{i \neq j} x_i y_j$$

$$= \sum_{i=1}^{n} x_i y_j + \sum_{i<j} x_i y_j + x_j y_i$$

$$\leq \sum_{i=1}^{n} x_i y_j + \sum_{i<j} x_i y_i + x_j y_j$$

$$= \sum_{i=1}^{n} x_i y_j + (n-1) \sum_{i=1}^{n} x_i y_j$$

$$= n \sum_{i=1}^{n} x_i y_j$$

4. Find the volume of a tetrahedron that has vertices at (1,0,0), (0,1,0), (0,0,1), (1,1,1).

Solution: Note that the distance between any pair of these points is $\sqrt{2}$. Thus this is an equilateral tetrahedron. As a type of pyramid, its volume is Bh/3, where B is the area of the base and h is the height. B is just the area of an equilateral triangle with side length $\sqrt{2}$; so B is $\sqrt{3}/2$. The height h is the distance from (1,1,1) to the plane determined by the other 3 points, which has equation $x+y+z=1$. The formula for the distance from point (a',b',c') to the plane $ax+by+cz+d=0$ is $\dfrac{|aa'+bb'+cc'+d|}{\sqrt{a^2+b^2+c^2}}$ which gives the height as $2/\sqrt{3}$. (Alternatively, one could note that the apex of tetrahedron, the centroid of the base, and the midpoint of a side of the base form a right triangle. The hypotenuse of this triangle is the slant-height is also the altitude of one face, which is $\sqrt{6}/2$. The distance from the side to the centroid is 1/3 the altitude of the base, which is $\sqrt{6}/6$. The Pythagorean Theorem then gives $h=2/\sqrt{3}$. Either way the volume is $\dfrac{1}{3}\dfrac{\sqrt{3}}{2}\dfrac{2}{\sqrt{3}} = \dfrac{1}{3}$.

Alternate Solution (for those familiar with vector calculus): Pick one of the vertices and determine the three vectors to the other three vertices. Call these vectors u, v, w. The volume of the tetrahedron is 1/6 that of the volume of the parallelopiped determined by these three vectors, which is given by the absolute value of the "scalar triple product," e.g. $|u \cdot (v \times w)|$. If we take (1,1,1) as our vertex, then u, v, w equal <0,-1,-1>, <-1,0,-1>, and <-1,-1,0> and the triple product is just the determinant of the matrix with these vectors as the rows. The absolute value of the determinant is 2, which implies the answer derived above, 1/3.

5. Ella and Daniel are preparing to play a game called "Twenty-One Stones". The 21 stones will be placed in 2 piles. Ella will alternate turns. On his or her turn, each player is to remove as many stones as he or she wants from either pile or remove the same number from both piles. The player to take the last stone wins. Ella says since she is younger, she gets to go first. Daniel says "fine – but I get to divide the stones into piles." How many stones should Daniel put into each pile so that he can guarantee that he wins the game?

Solution: Consider the set of safe positions for Daniel to leave Ella. Obviously (0,0) is safe, since it means Daniel has already won. What is the simplest position from which Ella could not win – clearly (1,2) – this is the next safe position. What is the next position up from which Ella could not leave Daniel with (1,2). This is (3,5). After this comes (4,7), (6,10), and (8,13). The pattern would continue with each pair having a difference one greater than the previous pair and each first number being one or two greater than the previous first number – depending on whether or not the number one greater was already part of a safe pair or not. So, we could continue, but we already see the answer is Daniel should make one pile of 8 stones and another pile of 13. To double check, if he picked (10,11) or (9,12), Ella could in one move go all the way to (1,2) or (4,7), respectively. Everyone of Daniel's other choices involves leaving one pile of 7 or lower and Ella likewise could, in a single move, go to one of the safe positions and force a win for herself.

6. If a, b,c are the side lengths of a triangle, prove that
$$abc \geq (a+b-c)(b+c-a)(c+a-b)$$

Solution: A common substitution in triangle problems is to let
$a = x+y,\ b = y+z,\ c = z+x$. One reason this is often helpful is that not all possible triplets of values for a, b, c correspond to a triangle, but any non-negative choices for x, y, and z do correspond to a triangle. In any case, with this (invertible) substitution, the given inequality is equivalent to
$(x+y)(y+z)(z+x) \geq 2y \cdot 2z \cdot 2x$. This follows directly from the AM-GM inequality, which implies that $x+y \geq 2\sqrt{xy}$, $y+z \geq 2\sqrt{yz}$, $z+x \geq 2\sqrt{zx}$.