Operators and Operator Algebras in Quantum Mechanics

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Outline

- v Quantum Harmonic Oscillator
- Ø Schrödinger Equation (SE)

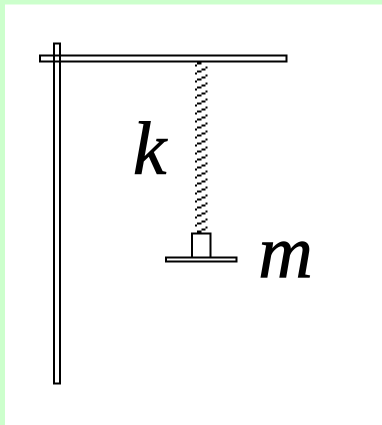
  Ladder Operators
- Ø Coherent and Squeezed Oscillator States

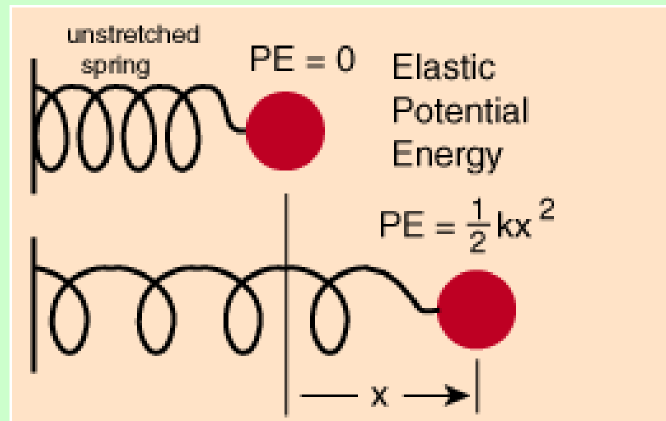
Classical Harmonic Oscillator

Total Energy = Kinetic Energy + Potential Energy

$$H = K + V(x) = \frac{p^2}{2m} + \frac{mw^2x^2}{2}$$

Hamiltonian Function





Frequency

$$w = \sqrt{\frac{k}{m}} = \frac{2p}{T}$$

Period

Total Energy: $E = mw^2A^2$

Amplitude

Quantum Harmonic Oscillator

Total Energy = Kinetic Energy + Potential Energy

$$\hat{H} = \hat{K} + \hat{V} = \frac{\hat{p}^2}{2m} + \frac{mw^2\hat{x}^2}{2}$$

Hamiltonian Operator:
Acts on Wave Functions

$$\Psi = \Psi(x,t) = \langle x | \Psi \rangle$$

Probability Density:

$$|\Psi(x,t)|^2$$

Momentum Operator $\hat{p} = -i\hbar\frac{\partial}{\partial x}$

Coordinate Operator $\hat{x} = x$

Time-Dependent Schroedinger Equation:

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = \hat{H}\Psi(x,t)$$

Erwin Schrödinger

Once at the end of a colloquium I heard Debye saying something like: "Schrödinger, you are not working right now on very important problems… why don't you tell us some time about that thesis of de Broglie's…

In one of the next colloquia, Schrödinger gave a beautifully clear account of how de Broglie associated a wave with a particle, and how he could obtain the quantization rules…

When he had finished, Debye casually remarked that he thought this way of talking was rather childish… To deal properly with waves, one had to have a wave equation.

Felix Bloch
Address to the American Physical Society, 1976

SE as an Eigenvalue Equation

Time-Dependent Schroedinger Equation (SE):

$$i\mathbf{h}\frac{\partial\Psi(x,t)}{\partial t} = \hat{H}\Psi(x,t) = E\Psi(x,t)$$

Energy Eigenvalue E

$$\Psi(x,t) = j\,(x)\exp(-i\frac{E}{\mathbf{h}}t) := j\,(x)\exp(-iwt)$$

Time-Independent SE:

$$\hat{H}j\,(x) = Ej\,(x)$$

For a Harmonic Oscillator: $$-\frac{\mathbf{h}^2}{2m}\frac{d^2 j}{dx^2} + \frac{mw^2x^2}{2}j = Ej$$

SE for Quantum Harmonic Oscillator

$$\frac{d^2 j}{dx^2} + \frac{2m}{\mathbf{h}^2}\left(E - \frac{mw^2 x^2}{2} \right) j = 0 \qquad -\infty < x < +\infty$$

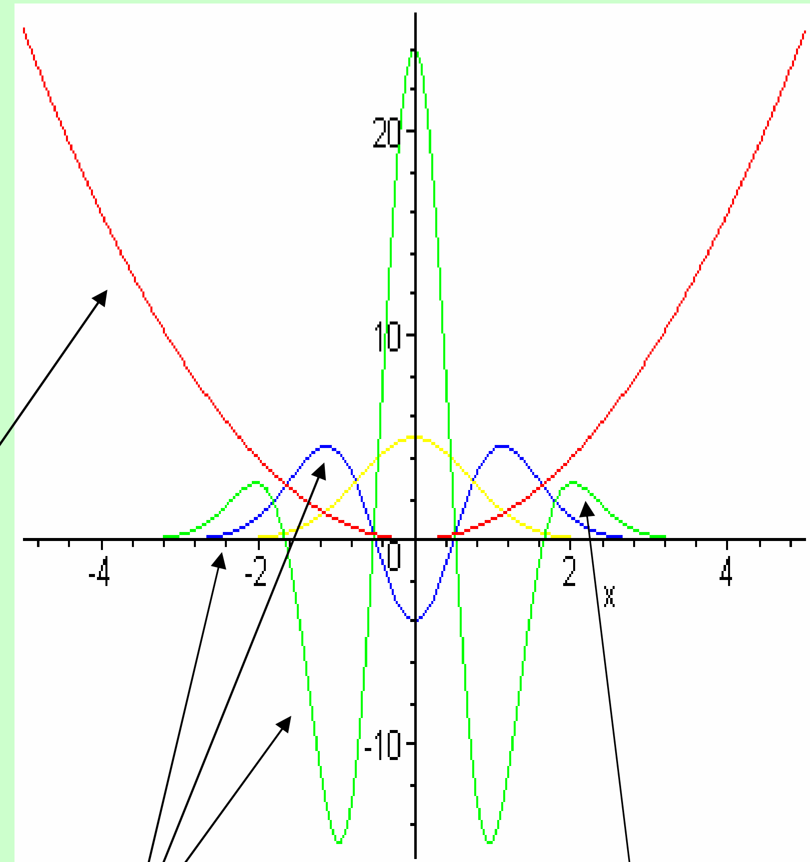
- **A well-known equation in mathematics**
- **Solutions were explored by Charles Hermite (1822-1901)**

Requirements for a Solution

Physically reasonable wave functions must drop to zero for $|x| \rightarrow \infty$

Symmetry of the potential => even and odd solutions

Discrete Energy Eigenvalues E_n are only allowed



Potential Barrier

Eigenfunctions: QM Wavefunctions

QM: Penetration "Below the Barrier"

(Quantum) Operator Approach: Non-Commutativity, Ladder Operators, …

Non-Commutative Algebras

Momentum Operator $\hat{p} = -i\hbar \frac{\partial}{\partial x}$

Coordinate Operator $\hat{x} = x$

Commutator: $[\hat{p}, \hat{x}] = \hat{p}\hat{x} - \hat{x}\hat{p} = -i\hbar$

$$[\hat{p}, \hat{x}] f = -i\hbar \frac{d}{dx}(fx) + i\hbar \frac{df}{dx} x = -i\hbar f \quad \forall \ f(x)$$

Non-Commutativity =>
Heisenberg Uncertainty Principle

Hermitian and Unitary Operators

Hermitian Conjugates:

$$\langle y \,|\hat{A}|\, j \rangle = \left(\langle j \,|\hat{A}^{+}|\, y \rangle\right)^{*}$$

or, in coordinate representation,

$$\int dx\, y(x)^{*} \hat{A}(x) j(x) = \left(\int dx\, j(x)^{*} \hat{A}^{+}(x) y(x)\right)^{*}$$

Hermitian Operators:

$$\hat{A} = \hat{A}^{+}$$

Real Eigenvalues
Complete Set of Eigenstates

Physical Observables in QM are described by Hermitian Operators

Unitary Operators:

$$\hat{U}^{+} = \hat{U}^{-1}$$

Describe Time Evolution in QM
Perform Operator Transformations

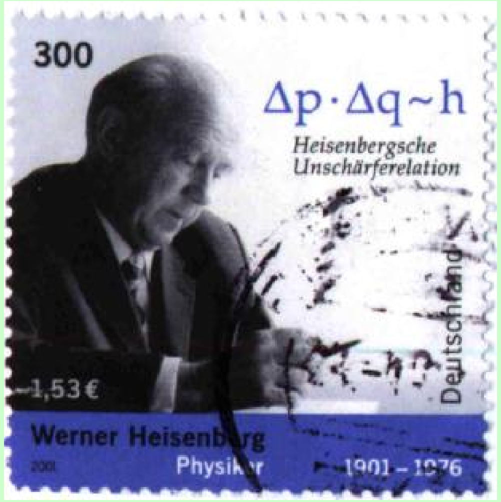
$$\hat{A} \rightarrow \hat{B} = \hat{U}\hat{A}\hat{U}^{+}$$

Heisenberg Uncertainty Principle

Non-Commutativity: $[\hat{p}, \hat{x}] = \hat{p}\hat{x} - \hat{x}\hat{p} = -i\mathbf{h}$

Heisenberg Uncertainty Principle:

$$\Delta p \cdot \Delta x \geq \mathbf{h}/2$$



$$\Delta A = \sqrt{\left\langle \hat{A}^2 \right\rangle - \left\langle \hat{A} \right\rangle^2}$$

Uncertainty= Mean Square Deviation
= Sqrt[Variance]

$$\left\langle \hat{A} \right\rangle = \left\langle j \left| \hat{A} \right| j \right\rangle = \int dx\, j(x)^* \hat{A}(x) j(x)$$

Expectation Value of A

Physical Observable A   => Hermitian (self-adjoint) operator $\hat{A}$

Non-Commutative Ladder Operators

Dimensionless Operators

$$\hat{X} = \sqrt{\frac{mw}{\mathbf{h}}}\hat{x} = X$$

$$\hat{P} = \frac{\hat{p}}{\sqrt{mw\mathbf{h}}} = -i\frac{d}{dX}$$

$$[\hat{P},\hat{X}] = -i$$

Ladder Operators

$$a = \frac{1}{\sqrt{2}}\left(\hat{X} + i\hat{P}\right) = \frac{1}{\sqrt{2}}\left(X + \frac{d}{dX}\right)$$

$$a^{+} = \frac{1}{\sqrt{2}}\left(\hat{X} - i\hat{P}\right) = \frac{1}{\sqrt{2}}\left(X - \frac{d}{dX}\right)$$

Non-commutative algebra:

$$[a, a^{+}] = 1$$

Hamiltonian and Ladder Operators

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{mw^2\hat{x}^2}{2} = \mathbf{h}w\left(a^+a + \tfrac{1}{2}\right)$$

algebra: $[a, a^+] = 1$

Lowering ***Raising***

$$a|0\rangle = 0$$

Vacuum State

$$a = \frac{1}{\sqrt{2}}\left(X + \frac{d}{dX}\right)$$

Coordinate Representation:

$$\langle X|0\rangle = \exp\left(-\tfrac{1}{2}X^2\right)$$

Gaussian

(Bose) Ladder Operators

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(a^{+}\right)^{n}|0\rangle$$

Excited States n = 1, 2, 3, …

algebra

$$[a, a^{+}] = 1$$

=>

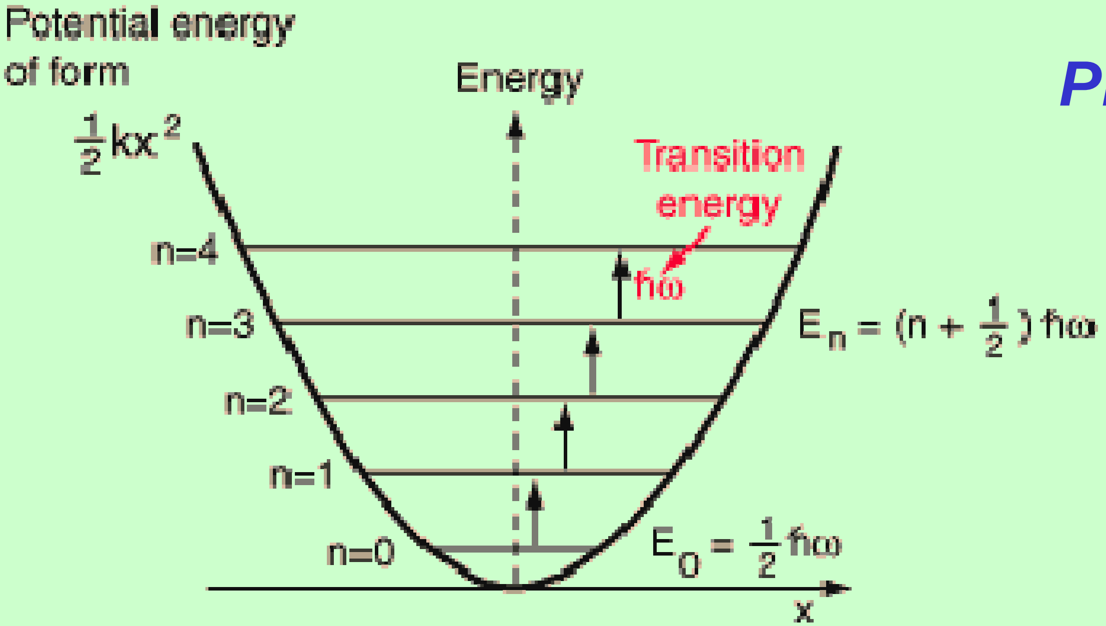
Raising $$a^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

Lowering $$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

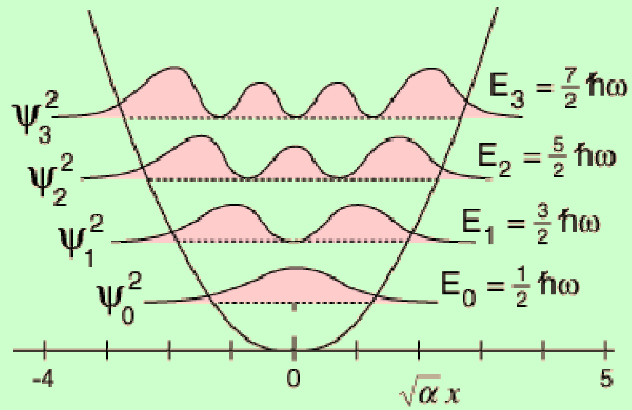
Particle Number Operator $$a^{+}a|n\rangle = n|n\rangle$$

Hamiltonian, Ladder Operators, Eigenstates

Equidistant Energy Levels



Probability Distributions



$$\hat{H} = \mathbf{h}w\left(a^{+}a + \tfrac{1}{2}\right)$$

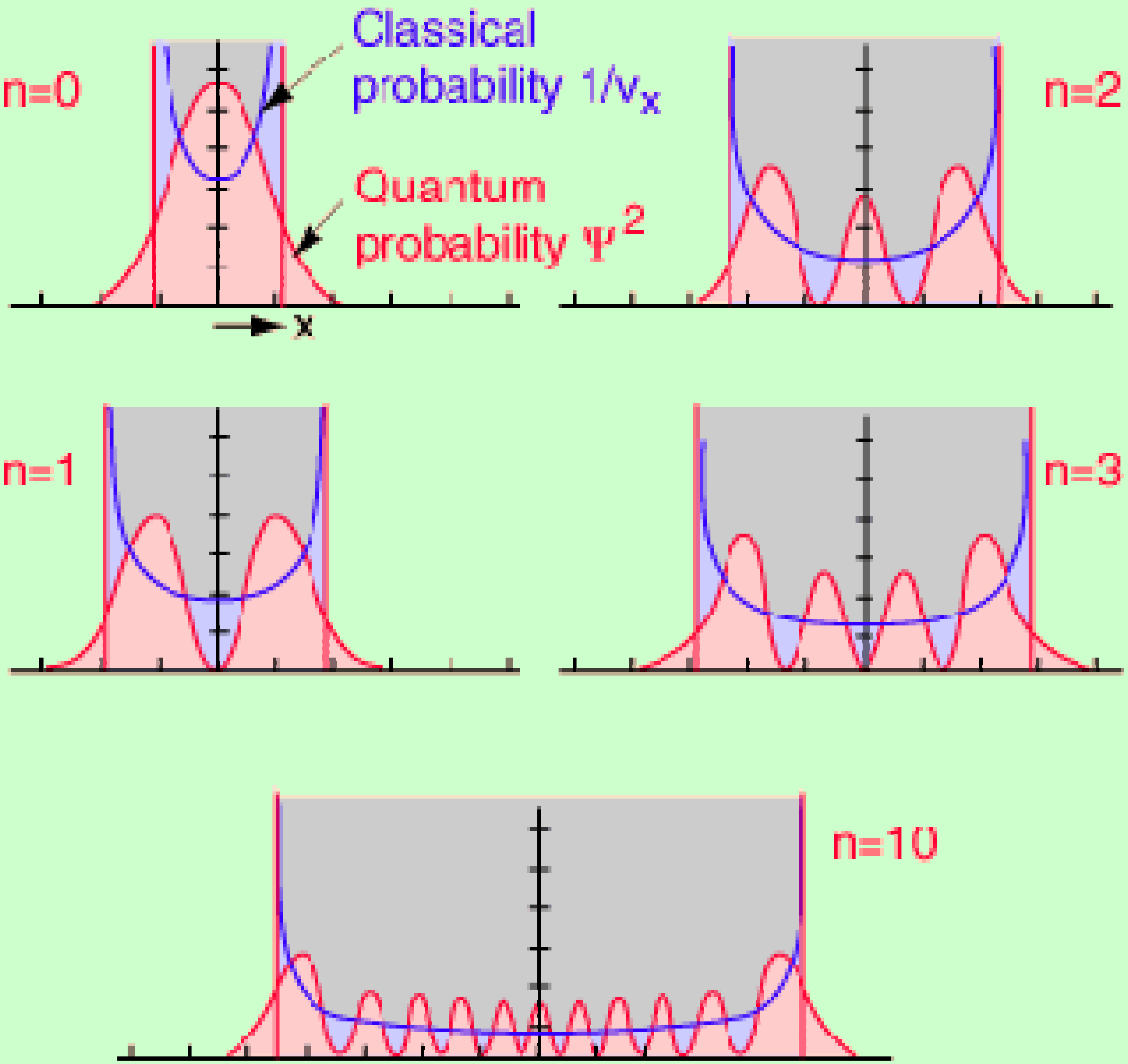
Hamiltonian (Energy Operator)

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(a^{+}\right)^{n}|0\rangle$$

n = 0, 1, 2, …

http://vega.fjfi.cvut.cz/docs/pvok/aplety/kv_osc/index.html

Oscillator Eigenstates: Probabilty Distribution



http://hyperphysics.phy-astr.gsu.edu/hbase/quantum/

Oscillator Coherent States

Displacement Operator $$\hat{D}(a) = \exp\left(a a^{+} - a^{*} a\right)$$

Complex number

Unitary

Coherent States $$\left|a\right\rangle = \hat{D}(a)\left|0\right\rangle$$

Schrödinger 1927
(in a different form)

Minimum Uncertainty States

$$\Delta p \cdot \Delta x = \hbar / 2$$

Behave "most classically"

$$\left|a\right\rangle = \exp\left(-\tfrac{1}{2}|a|^2\right)\sum_{n=0}^{\infty}\frac{a^n}{\sqrt{n!}}\left|n\right\rangle$$

Disentangling the Operators

Displacement Operator

$$\hat{D}(a) = \exp\left(a a^{+} - a^{*} a\right) =$$
$$= \exp\left(-\tfrac{1}{2}|a|^{2}\right)\exp\left(a a^{+}\right)\exp\left(-a^{*} a\right)$$

Baker-Campbell-Hausdorf formula

Algebra Used: $[a, a^{+}] = 1$

Coherent States $|a\rangle = \hat{D}(a)|0\rangle$

$$|a\rangle = \exp\left(-\tfrac{1}{2}|a|^{2}\right)\sum_{n=0}^{\infty}\frac{a^{n}}{\sqrt{n!}}|n\rangle$$

Superposition of an infinite # of oscillator states |n> with proper amplitudes and phases

Displacement Operator

$$\hat{D}(a) = \exp\left(a a^{+} - a^{*} a\right)$$

$$\hat{D}^{+}(a) a^{+} \hat{D}(a) = a^{+} + a$$

$$\hat{D}^{+}(a) a \hat{D}(a) = a + a^{*}$$

Generates an Over<u>complete</u> Set of Coherent States

$$\int \frac{d\,\mathrm{Re}\,a\; d\,\mathrm{Im}\,a}{p} \left| a \right\rangle \left\langle a \right| = \hat{1}$$

Very useful in Quantum Optics and Field Theory

$$\left| a \right\rangle = \hat{D}(a) \left| 0 \right\rangle$$

Squeezing Operator

$$\hat{S}(b) = \exp\left(\tfrac{1}{2} b a^{+2} - \tfrac{1}{2} b^* a^2\right)$$

Unitary

Mixes Creation and Annihilation Operators:

$$\hat{S}^+(b) a^+ \hat{S}(b) = \cosh(b) a^+ - \sinh(b) a$$

$$\hat{S}^+(b) a \hat{S}(b) = \cosh(b) a - \sinh(b) a^+$$

Generates Squeezed States: $|b\rangle = \hat{S}(b)|0\rangle$

Minimum Uncertainty States + One coordinate uncertainty E.g., Δx <u>can be made arbitrarily small</u> (squeezed) at the expense of Δp. **Reduction of Quantum Noise**

Bose and Fermi Statisics

Bose-Einstein Statistics: Commutation Relation

$$[a, a^+] = aa^+ - a^+a = 1$$

Bosons: photons, phonons, …

Squeezed States: Theory of Superfluidity

Fermi-Dirac Statistics: AntiCommutation Relation

$$\{a, a^+\} = aa^+ + a^+a = 1$$

Fermions: electrons, protons, …

Squeezed States: Theory of Superconductivity

How successful is quantum mechanics?

It works nicely and not just for tiny particles

QM explains Macroscopic Quantum Phenomena: Superconductivity, Superfluidity, Josephson Effect, and the Quantum Hall Effect

QM is crucial to explain how the Universe formed
QM "explains" the stability of matter

No deviations from quantum mechanics are known

A HUGE problem: to describe gravitation in the framework of quantum mechanics

"Theory of everything" will be a relativistic quantum-mechanical theory