Chapter 4: Randomized Blocks and Latin Squares

Design of Engineering Experiments – The Blocking Principle

- **Blocking** and **nuisance factors**
- The randomized complete block design or the **RCBD**
- Extension of the ANOVA to the RCBD
- Other blocking scenarios…Latin square designs

The Blocking Principle

- **Blocking** is a technique for dealing with **nuisance factors**
- A **nuisance** factor is a factor that probably has some effect on the response, but it's of no interest to the experimenter…however, the variability it transmits to the response needs to be minimized
- Typical nuisance factors include batches of raw material, operators, pieces of test equipment, time (shifts, days, etc.), different experimental units
- **Many** industrial experiments involve blocking (or should)
- Failure to block is a common flaw in designing an experiment (consequences?)

The Blocking Principle

- If the nuisance variable is **known** and **controllable**, we use **blocking**
- If the nuisance factor is **known** and **uncontrollable**, sometimes we can use the **analysis of covariance** (see Chapter 15) to remove the effect of the nuisance factor from the analysis
- If the nuisance factor is **unknown** and **uncontrollable** (a **"lurking" variable**), we hope that **randomization** balances out its impact across the experiment
- Sometimes several sources of variability are **combined** in a block, so the block becomes an aggregate variable

The Hardness Testing Example

We wish to determine whether 4 different tips produce different (mean) hardness reading on a Rockwell hardness tester. The machine operates by pressing the tip into a metal test coupon, and from the depth of the resulting depression, the hardness of the coupon can be determined. Take 4 observations for each tip.

- Assignment of the tips to an **experimental unit**; that is, a test coupon
- The test coupons are a source of **nuisance variability** **(heat)**
- Completed randomized experiment: the experiment error will reflect both random error and variability between coupons.
- Alternatively, the experimenter may want to test the tips across coupons of various hardness levels
- Randomized complete block design (RCBD): remove the variability between coupons by testing each tip once on each of the four coupons. (see Table 4.1)
- "Complete" indicates that each **block** (coupon) contains all the **treatments** (tips)

The Hardness Testing Example

- To conduct this experiment as a RCBD, assign all 4 tips to each coupon
- Each coupon is called a "**block**"; that is, it's a more homogenous experimental unit on which to test the tips
- Variability **between** blocks can be large, variability **within** a block should be relatively small
- In general, a **block** is a specific level of the nuisance factor
- A complete replicate of the basic experiment is conducted in each block
- A block represents a **restriction on randomization**
- All runs **within** a block are **randomized**

The Hardness Testing Example

- Suppose that we use b = 4 blocks:

■ **TABLE 4.1**
Randomized Complete Block Design for the Hardness Testing Experiment

Test Coupon (Block)			
1	**2**	**3**	**4**
Tip 3	Tip 3	Tip 2	Tip 1
Tip 1	Tip 4	Tip 1	Tip 4
Tip 4	Tip 2	Tip 3	Tip 2
Tip 2	Tip 1	Tip 4	Tip 3

- Notice the **two-way structure** of the experiment
- Once again, we are interested in testing the equality of treatment means, but now we have to remove the variability associated with the nuisance factor (the blocks)

Extension of the ANOVA to the RCBD

- Suppose that there are *a* treatments (factor levels) and *b* blocks
- A **statistical model** (effects model) for the RCBD is

$$y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \end{cases}$$

- The relevant (fixed effects) hypotheses are

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_a \text{ where } \mu_i = (1/b)\sum_{j=1}^{b}(\mu + \tau_i + \beta_j) = \mu + \tau_i$$

Extension of the ANOVA to the RCBD

ANOVA partitioning of total variability:

$$\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\bar{y}_{..})^2 = \sum_{i=1}^{a}\sum_{j=1}^{b}[(\bar{y}_{i.}-\bar{y}_{..})+(\bar{y}_{.j}-\bar{y}_{..})$$

$$+(y_{ij}-\bar{y}_{i.}-\bar{y}_{.j}+\bar{y}_{..})]^2$$

$$= b\sum_{i=1}^{a}(\bar{y}_{i.}-\bar{y}_{..})^2 + a\sum_{j=1}^{b}(\bar{y}_{.j}-\bar{y}_{..})^2$$

$$+\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\bar{y}_{i.}-\bar{y}_{.j}+\bar{y}_{..})^2$$

$$SS_T = SS_{Treatments} + SS_{Blocks} + SS_E$$

Extension of the ANOVA to the RCBD

The degrees of freedom for the sums of squares in

$$SS_T = SS_{Treatments} + SS_{Blocks} + SS_E$$

are as follows:

$$ab - 1 = a - 1 + b - 1 + (a-1)(b-1)$$

Therefore, ratios of sums of squares to their degrees of freedom result in mean squares and the ratio of the mean square for treatments to the error mean square is an *F* statistic that can be used to test the hypothesis of equal treatment means

ANOVA Display for the RCBD

■ **TABLE 4.2**

Analysis of Variance for a Randomized Complete Block Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$SS_{\text{Treatments}}$	$a - 1$	$\dfrac{SS_{\text{Treatments}}}{a-1}$	$\dfrac{MS_{\text{Treatments}}}{MS_E}$
Blocks	SS_{Blocks}	$b - 1$	$\dfrac{SS_{\text{Blocks}}}{b-1}$	
Error	SS_E	$(a-1)(b-1)$	$\dfrac{SS_E}{(a-1)(b-1)}$	
Total	SS_T	$N - 1$		

Manual computing:

$$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{b} y_{ij}^2 - \frac{y_{..}^2}{N} \quad \textbf{(4.9)}$$

$$SS_{\text{Treatments}} = \frac{1}{b} \sum_{i=1}^{a} y_{i.}^2 - \frac{y_{..}^2}{N} \quad \textbf{(4.10)}$$

$$SS_{\text{Blocks}} = \frac{1}{a} \sum_{j=1}^{b} y_{.j}^2 - \frac{y_{..}^2}{N} \quad \textbf{(4.11)}$$

and the error sum of squares is obtained by subtraction as

$$SS_E = SS_T - SS_{\text{Treatments}} - SS_{\text{Blocks}} \quad \textbf{(4.12)}$$

EXAMPLE 4.1

A medical device manufacturer produces vascular grafts (artificial veins). These grafts are produced by extruding billets of polytetrafluoroethylene (PTFE) resin combined with a lubricant into tubes. Frequently, some of the tubes in a production run contain small, hard protrusions on the external surface. These defects are known as "flicks." The defect is cause for rejection of the unit.

The product developer responsible for the vascular grafts suspects that the extrusion pressure affects the occurrence of flicks and therefore intends to conduct an experiment to investigate this hypothesis. However, the resin is manufactured by an external supplier and is delivered to the medical device manufacturer in batches. The engineer also suspects that there may be significant batch-to-batch variation, because while the material should be consistent with respect to parameters such as molecular weight, mean particle size, retention, and peak height ratio, it probably isn't due to manufacturing variation at the resin supplier and natural variation in the material. Therefore, the product developer decides to investigate the effect of four different levels of extrusion pressure on flicks using a randomized complete block design considering batches of resin as blocks. The RCBD is shown in Table 4.3. Note that there are four levels of extrusion pressure (treatments) and six batches of resin (blocks). Remember that the order in which the extrusion pressures are tested within each block is random. The response variable is yield, or the percentage of tubes in the production run that did not contain any flicks.

Vascular Graft Example (pg. 145)

- To conduct this experiment as a RCBD, assign all 4 pressures to each of the 6 batches of resin
- Each batch of resin is called a "**block**"; that is, it's a more homogenous experimental unit on which to test the extrusion pressures

■ **TABLE 4.3**
Randomized Complete Block Design for the Vascular Graft Experiment

Extrusion Pressure (PSI)	Batch of Resin (Block) 1	2	3	4	5	6	Treatment Total
8500	90.3	89.2	98.2	93.9	87.4	97.9	556.9
8700	92.5	89.5	90.6	94.7	87.0	95.8	550.1
8900	85.5	90.8	89.6	86.2	88.0	93.4	533.5
9100	82.5	89.5	85.6	87.4	78.9	90.7	514.6
Block Totals	350.8	359.0	364.0	362.2	341.3	377.8	$y_{..} = 2155.1$

To perform the analysis of variance, we need the following sum of squares:

$$SS_T = \sum_{i=1}^{4}\sum_{j=1}^{6} y_{ij}^2 - \frac{y_{..}^2}{N}$$

$$= 193{,}999.31 - \frac{(2155.1)^2}{24} = 480.31$$

$$SS_{\text{Treatments}} = \frac{1}{b}\sum_{i=1}^{4} y_{i.}^2 - \frac{y_{..}^2}{N}$$

$$= \frac{1}{6}[(556.9)^2 + (550.1)^2 + (533.5)^2$$

$$+ (514.6)^2] - \frac{(2155.1)^2}{24} = 178.17$$

$$SS_{\text{Blocks}} = \frac{1}{a}\sum_{j=1}^{6} y_{.j}^2 - \frac{y_{..}^2}{N}$$

$$= \frac{1}{4}[(350.8)^2 + (359.0)^2 + \cdots + (377.8)^2]$$

$$- \frac{(2155.1)^2}{24} = 192.25$$

$$SS_E = SS_T - SS_{\text{Treatments}} - SS_{\text{Blocks}}$$

$$= 480.31 - 178.17 - 192.25 = 109.89$$

The ANOVA is shown in Table 4.4. Using $\alpha = 0.05$, the critical value of F is $F_{0.05,3,15} = 3.29$. Because $8.11 > 3.29$, we conclude that extrusion pressure affects the mean yield. The P-value for the test is also quite small. Also, the resin batches (blocks) seem to differ significantly, because the mean square for blocks is large relative to error.

Vascular Graft Example

Response: Yield
ANOVA for Selected Factorial Model
Analysis of Variance Table [Partial Sum of Squares]

Source	Sum of Squares	DF	Mean Square	*F* Value	Prob > *F*
Block	192.25	5	38.45		
Model	178.17	3	59.39	8.11	0.0019
A	*178.17*	*3*	*59.39*	*8.11*	*0.0019*
Residual	109.89	15	7.33		
Cor Total	480.31	23			

Std. Dev.	2.71	R-Squared	0.6185
Mean	89.80	Adj R-Squared	0.5422
C.V.	3.01	Pred R-Squared	0.0234
PRESS	281.31	Adeq Precision	9.759

Treatment Means (Adjusted, If Necessary)

	Estimated Mean	Standard Error
1–8500	92.82	1.10
2-8700	91.68	1.10
3-8900	88.92	1.10
4-9100	85.77	1.10

Treatment	Mean Difference	DF	Standard Error	*t* for H_0 Coeff=0	Prob > |*t*|
1 vs.2	1.13	1	1.56	0.73	0.4795
1 vs.3	3.90	1	1.56	2.50	0.0247
1 vs.4	7.05	1	1.56	4.51	0.0004
2 vs.3	2.77	1	1.56	1.77	0.0970
2 vs.4	5.92	1	1.56	3.79	0.0018
3 vs.4	3.15	1	1.56	2.02	0.0621

Diagnostics Case Statistics

Standard Order	Actual Value	Predicted Value	Residual	Leverage	Student Residual	Cook's Distance	Outlier *t*	Run Order
1	90.30	90.72	–0.42	0.375	–0.197	0.003	–0.190	1
2	89.20	92.77	–3.57	0.375	–1.669	0.186	–1.787	6
3	98.20	94.02	4.18	0.375	1.953	0.254	2.185	9
4	93.90	93.57	0.33	0.375	0.154	0.002	0.149	13
5	87.40	88.35	–0.95	0.375	–0.442	0.013	–0.430	19
6	97.90	97.47	0.43	0.375	0.201	0.003	0.194	23
7	92.50	89.59	2.91	0.375	1.361	0.124	1.405	4
8	89.50	91.64	–2.14	0.375	–0.999	0.067	–0.999	5
9	90.60	92.89	–2.29	0.375	–1.069	0.076	–1.075	10
10	94.70	92.44	2.26	0.375	1.057	0.075	1.062	16
11	87.00	87.21	–0.21	0.375	–0.099	0.001	–0.096	20
12	95.80	96.34	–0.54	0.375	–0.251	0.004	–0.243	21
13	85.50	86.82	–1.32	0.375	–0.617	0.025	–0.604	3
14	90.80	88.87	1.93	0.375	0.902	0.054	0.896	8
15	89.60	90.12	–0.52	0.375	–0.243	0.004	–0.236	12
16	86.20	89.67	–3.47	0.375	–1.622	0.175	–1.726	15
17	88.00	84.45	3.55	0.375	1.661	0.184	1.776	17
18	93.40	93.57	–0.17	0.375	–0.080	0.000	–0.077	22
19	82.50	83.67	–1.17	0.375	–0.547	0.020	–0.534	2
20	89.50	85.72	3.78	0.375	1.766	0.208	1.917	7
21	85.60	86.97	–1.37	0.375	–0.641	0.027	–0.628	11
22	87.40	86.52	0.88	0.375	0.411	0.011	0.399	14
23	78.90	81.30	–2.40	0.375	–1.120	0.084	–1.130	18
24	90.70	90.42	0.28	0.375	0.130	0.001	0.126	24

Oneway Analysis of Yield By Pressure
Block
Batch

Oneway Anova
Summary of Fit

Rsquare	0.771218
Adj Rsquare	0.649201
Root Mean Square Error	2.706612
Mean of Response	89.79583
Observations (or Sum Wgts)	24

Analysis of Variance

Source	DF	Sum of Squares	Mean Square	*F* Ratio	Prob > F
Pressure	3	178.17125	59.3904	8.1071	0.0019
Batch	5	192.25208	38.4504	5.2487	0.0055
Error	15	109.88625	7.3257		
C.Total	23	480.30958			

Means for Oneway Anova

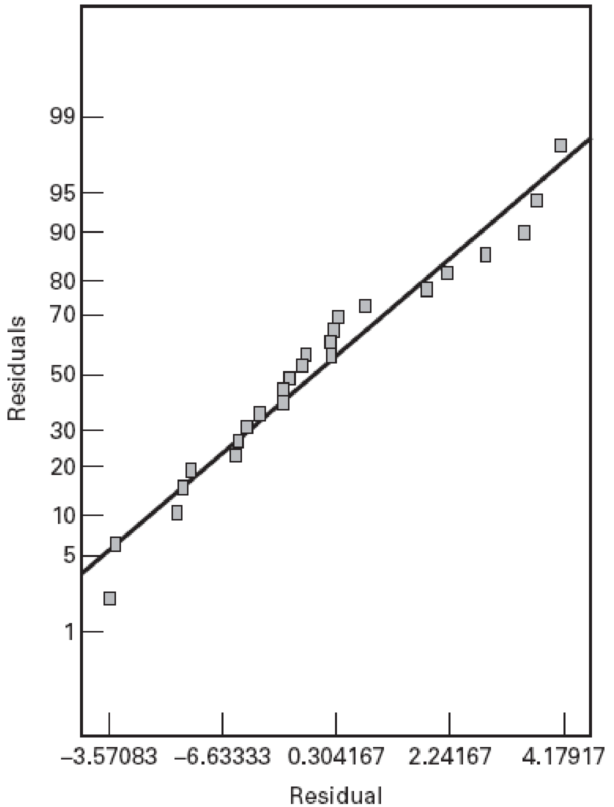
Level	Number	Mean	Std. Error	Lower 95%	Upper 95%
8500	6	92.8167	1.1050	90.461	95.172
8700	6	91.6833	1.1050	89.328	94.039
8900	6	88.9167	1.1050	86.561	91.272
9100	6	85.7667	1.1050	83.411	88.122

Std. Error uses a pooled estimate of error variance

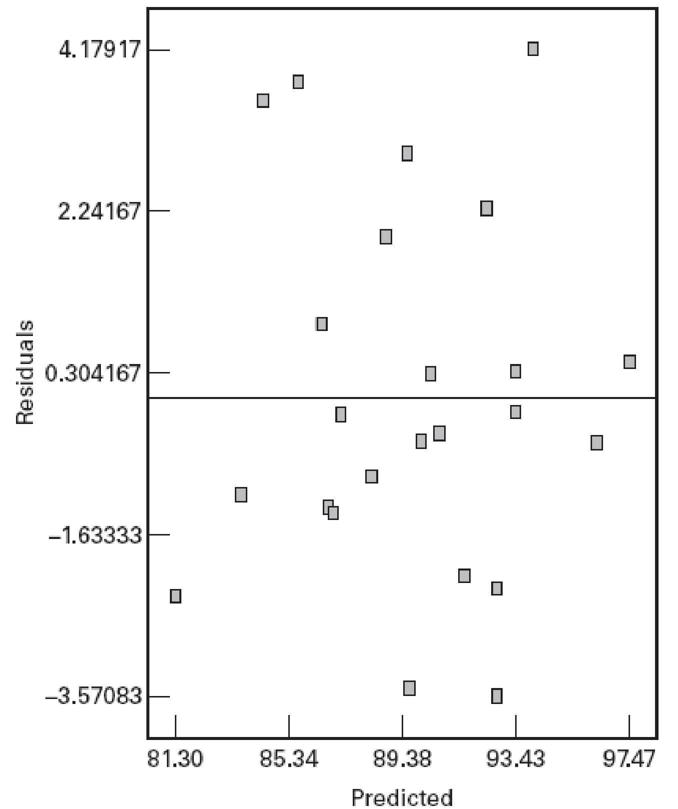
Block Means

Batch	Mean	Number
1	87.7000	4
2	89.7500	4
3	91.0000	4
4	90.5500	4
5	85.3250	4
6	94.4500	4

Residual Analysis for the Vascular Graft Example

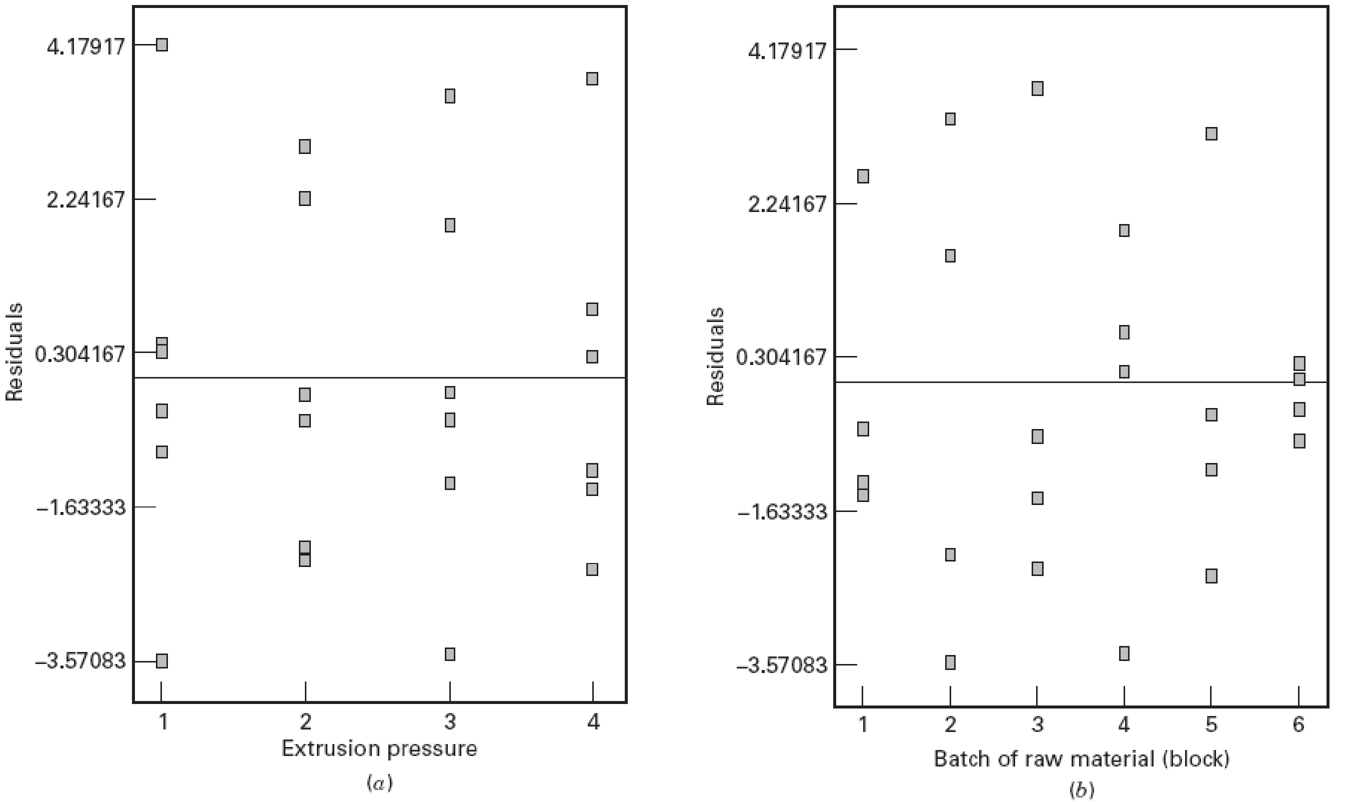


■ **FIGURE 4.4** Normal probability plot of residuals for Example 4.1



■ **FIGURE 4.5** Plot of residuals versus $\hat{y}_{ij}$ for Example 4.1

Residual Analysis for the Vascular Graft Example



■ **FIGURE 4.6** Plot of residuals by extrusion pressure (treatment) and by batches of resin (block) for Example 4.1

Residual Analysis for the Vascular Graft Example

- Basic residual plots indicate that **normality**, **constant variance** assumptions are satisfied
- No obvious problems with **randomization**
- **No patterns in the residuals vs. block**
- Can also plot residuals versus the pressure (residuals by factor)
- These plots provide more information about the constant variance assumption, possible outliers

Multiple Comparisons for the Vascular Graft Example – Which Pressure is Different?

Treatment Means (Adjusted, If Necessary)

	Estimated Mean	Standard Error
1–8500	92.82	1.10
2-8700	91.68	1.10
3-8900	88.92	1.10
4-9100	85.77	1.10

Treatment	Mean Difference	DF	Standard Error	t for H_0 Coeff=0	Prob > |t|
1 vs 2	1.13	1	1.56	0.73	0.4795
1 vs 3	3.90	1	1.56	2.50	0.0247
1 vs 4	7.05	1	1.56	4.51	0.0004
2 vs 3	2.77	1	1.56	1.77	0.0970
2 vs 4	5.92	1	1.56	3.79	0.0018
3 vs 4	3.15	1	1.56	2.02	0.0621

Also see Figure 4.2, Design-Expert output

The Latin Square Design

- These designs are used to simultaneously control (or eliminate) **two sources of nuisance variability**
- A significant assumption is that the three factors (treatments, nuisance factors) **do not interact**
- If this assumption is violated, the Latin square design will not produce valid results
- Latin squares are not used as much as the RCBD in industrial experimentation

Example: Latin Square Design

- Suppose that an experimenter is studying the effects of five different formulations of a rocket propellant used in aircrew escape systems on the observed burning rate. Each formulation is mixed from a batch of raw material that only large enough for five formulations to be tested. Furthermore, the formulations are prepared by several operators, and there may be substantial differences in the skills and experience of the operators.
- Two nuisance factors: batches of raw material and operators.
- The appropriate design: testing each formulation exactly once in each batch of raw material and for each formulation to be prepared exactly once by each of the five operators.

The Rocket Propellant Problem – A Latin Square Design

■ **TABLE 4.9**
Latin Square Design for the Rocket Propellant Problem

Batches of Raw Material	Operators				
	1	2	3	4	5
1	$A = 24$	$B = 20$	$C = 19$	$D = 24$	$E = 24$
2	$B = 17$	$C = 24$	$D = 30$	$E = 27$	$A = 36$
3	$C = 18$	$D = 38$	$E = 26$	$A = 27$	$B = 21$
4	$D = 26$	$E = 31$	$A = 26$	$B = 23$	$C = 22$
5	$E = 22$	$A = 30$	$B = 20$	$C = 29$	$D = 31$

- This is a 5×5 Latin square design. What is the # of replications?
- Columns and rows represent two restrictions on randomization.
- Page 159 shows some other Latin squares
- Table 4-13 (page 162) contains properties of Latin squares
- Statistical analysis?

Statistical Analysis of the Latin Square Design

- The statistical (effects) model is

$$y_{ijk} = \mu + \alpha_i + \tau_j + \beta_k + \varepsilon_{ijk} \begin{cases} i = 1, 2, \ldots, p \\ j = 1, 2, \ldots, p \\ k = 1, 2, \ldots, p \end{cases}$$

- The statistical analysis (ANOVA) is much like the analysis for the RCBD.
- See the ANOVA table, page 160 (Table 4.10)
- The analysis for the rocket propellant example follows

■ **TABLE 4.10**

Analysis of Variance for the Latin Square Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$SS_{\text{Treatments}} = \frac{1}{p}\sum_{j=1}^{p} y_{.j.}^2 - \frac{y_{...}^2}{N}$	$p - 1$	$\frac{SS_{\text{Treatments}}}{p-1}$	$F_0 = \frac{MS_{\text{Treatments}}}{MS_E}$
Rows	$SS_{\text{Rows}} = \frac{1}{p}\sum_{i=1}^{p} y_{i..}^2 - \frac{y_{...}^2}{N}$	$p - 1$	$\frac{SS_{\text{Rows}}}{p-1}$	
Columns	$SS_{\text{Columns}} = \frac{1}{p}\sum_{k=1}^{p} y_{..k}^2 - \frac{y_{...}^2}{N}$	$p - 1$	$\frac{SS_{\text{Columns}}}{p-1}$	
Error	SS_E (by subtraction)	$(p-2)(p-1)$	$\frac{SS_E}{(p-2)(p-1)}$	
Total	$SS_T = \sum_i \sum_j \sum_k y_{ijk}^2 - \frac{y_{...}^2}{N}$	$p^2 - 1$		

■ **TABLE 4.12**

Analysis of Variance for the Rocket Propellant Experiment

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-Value
Formulations	330.00	4	82.50	7.73	0.0025
Batches of raw material	68.00	4	17.00		
Operators	150.00	4	37.50		
Error	128.00	12	10.67		
Total	676.00	24			

As in any design problem, the experimenter should investigate the adequacy of the model by inspecting and plotting the residuals. For a Latin square, the residuals are given by

$$\begin{aligned} e_{ijk} &= y_{ijk} - \hat{y}_{ijk} \\ &= y_{ijk} - \bar{y}_{i..} - \bar{y}_{.j.} - \bar{y}_{..k} + 2\bar{y}_{...} \end{aligned}$$

Repeated Latin Squares: Case 1

TABLE 4.14

Analysis of Variance for a Replicated Latin Square, Case 1

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$\frac{1}{np}\sum_{j=1}^{p} y_{.j..}^2 - \frac{y_{....}^2}{N}$	$p-1$	$\frac{SS_{\text{Treatments}}}{p-1}$	$\frac{MS_{\text{Treatments}}}{MS_E}$
Rows	$\frac{1}{np}\sum_{i=1}^{p} y_{i...}^2 - \frac{y_{....}^2}{N}$	$p-1$	$\frac{SS_{\text{Rows}}}{p-1}$	
Columns	$\frac{1}{np}\sum_{k=1}^{p} y_{..k.}^2 - \frac{y_{....}^2}{N}$	$p-1$	$\frac{SS_{\text{Columns}}}{p-1}$	
Replicates	$\frac{1}{p^2}\sum_{l=1}^{n} y_{...l}^2 - \frac{y_{....}^2}{N}$	$n-1$	$\frac{SS_{\text{Replicates}}}{n-1}$	
Error	Subtraction	$(p-1)[n(p+1)-3]$	$\frac{SS_E}{(p-1)[n(p+1)-3]}$	
Total	$\sum\sum\sum\sum y_{ijkl}^2 - \frac{y_{....}^2}{N}$	np^2-1		

Table 4.14

Repeated Latin Squares: Case 2

■ **TABLE 4.15**

Analysis of Variance for a Replicated Latin Square, Case 2

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$\frac{1}{np}\sum_{j=1}^{p} y_{.j..}^2 - \frac{y_{....}^2}{N}$	$p-1$	$\frac{SS_{\text{Treatments}}}{p-1}$	$\frac{MS_{\text{Treatments}}}{MS_E}$
Rows	$\frac{1}{p}\sum_{l=1}^{n}\sum_{i=1}^{p} y_{i..l}^2 - \sum_{l=1}^{n}\frac{y_{...l}^2}{p^2}$	$n(p-1)$	$\frac{SS_{\text{Rows}}}{n(p-1)}$	
Columns	$\frac{1}{np}\sum_{k=1}^{p} y_{..k.}^2 - \frac{y_{....}^2}{N}$	$p-1$	$\frac{SS_{\text{Columns}}}{p-1}$	
Replicates	$\frac{1}{p^2}\sum_{l=1}^{n} y_{...l}^2 - \frac{y_{....}^2}{N}$	$n-1$	$\frac{SS_{\text{Replicates}}}{n-1}$	
Error	Subtraction	$(p-1)(np-1)$	$\frac{SS_E}{(p-1)(np-1)}$	
Total	$\sum_i\sum_j\sum_k\sum_l y_{ijkl}^2 - \frac{y_{....}^2}{N}$	np^2-1		

Table 4.15

Repeated Latin Squares: Case 3

■ **TABLE 4.16**

Analysis of Variance for a Replicated Latin Square, Case 3

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$\frac{1}{np}\sum_{j=1}^{p} y_{.j..}^2 - \frac{y_{....}^2}{N}$	$p-1$	$\frac{SS_{\text{Treatments}}}{p-1}$	$\frac{MS_{\text{Treatments}}}{MS_E}$
Rows	$\frac{1}{p}\sum_{l=1}^{n}\sum_{i=1}^{p} y_{i..l}^2 - \sum_{l=1}^{n}\frac{y_{...l}^2}{p^2}$	$n(p-1)$	$\frac{SS_{\text{Rows}}}{n(p-1)}$	
Columns	$\frac{1}{p}\sum_{l=1}^{n}\sum_{k=1}^{p} y_{..kl}^2 - \sum_{l=1}^{n}\frac{y_{...l}^2}{p^2}$	$n(p-1)$	$\frac{SS_{\text{Columns}}}{n(p-1)}$	
Replicates	$\frac{1}{p^2}\sum_{l=1}^{n} y_{...l}^2 - \frac{y_{....}^2}{N}$	$n-1$	$\frac{SS_{\text{Replicates}}}{n-1}$	
Error	Subtraction	$(p-1)[n(p-1)-1]$	$\frac{SS_E}{(p-1)[n(p-1)-1]}$	
Total	$\sum_i\sum_j\sum_k\sum_l y_{ijkl}^2 - \frac{y_{....}^2}{N}$	np^2-1		

Table 4.16

Graeco-Latin Squares

- Consider a p*p Latin square, and superimpose on it a second p*p Latin square in which the treatments are denoted by Greek letters. If the two squares when superimposed have the property that each Greek letter appears once and only once with each Latin letter, the two Latin squares are said to be ***orthogonal***, and the design obtained is called a ***Graeco-Latin square.***

■ **TABLE 4.18**

4 × 4 Graeco-Latin Square Design

	Column			
Row	**1**	**2**	**3**	**4**
1	$A\alpha$	$B\beta$	$C\gamma$	$D\delta$
2	$B\delta$	$A\gamma$	$D\beta$	$C\alpha$
3	$C\beta$	$D\alpha$	$A\delta$	$B\gamma$
4	$D\gamma$	$C\delta$	$B\alpha$	$A\beta$

Table 4.18

Graeco-Latin Squares

- **Example:** back to the rocket propellant example, suppose that we are also interested in the effect of test assemblies, which could be of importance. Let there be five test assemblies denoted by the Greek letters, α,β,γ,δ,and ε.
- Rows (raw material); columns (operators); Latin letters (formulations); Greek letters (test assemblies).

TABLE 4.20
Graeco-Latin Square Design for the Rocket Propellant Problem

Batches of Raw Material	Operators 1	2	3	4	5	$y_{i...}$
1	$A\alpha = -1$	$B\gamma = -5$	$C\epsilon = -6$	$D\beta = -1$	$E\delta = -1$	−14
2	$B\beta = -8$	$C\delta = -1$	$D\alpha = 5$	$E\gamma = 2$	$A\epsilon = 11$	9
3	$C\gamma = -7$	$D\epsilon = 13$	$E\beta = 1$	$A\delta = 2$	$B\alpha = -4$	5
4	$D\delta = 1$	$E\alpha = 6$	$A\gamma = 1$	$B\epsilon = -2$	$C\beta = -3$	3
5	$E\epsilon = -3$	$A\beta = 5$	$B\delta = -5$	$C\alpha = 4$	$D\gamma = 6$	7
$y_{..l}$	−18	18	−4	5	9	$10 = y_{...}$

Table 4.20

■ **TABLE 4.19**

Analysis of Variance for a Graeco-Latin Square Design

Source of Variation	Sum of Squares	Degrees of Freedom
Latin letter treatments	$SS_L = \frac{1}{p}\sum_{j=1}^{p} y_{.j..}^2 - \frac{y_{....}^2}{N}$	$p - 1$
Greek letter treatments	$SS_G = \frac{1}{p}\sum_{k=1}^{p} y_{..k.}^2 - \frac{y_{....}^2}{N}$	$p - 1$
Rows	$SS_{\text{Rows}} = \frac{1}{p}\sum_{i=1}^{p} y_{i...}^2 - \frac{y_{....}^2}{N}$	$p - 1$
Columns	$SS_{\text{Columns}} = \frac{1}{p}\sum_{l=1}^{p} y_{...l}^2 - \frac{y_{....}^2}{N}$	$p - 1$
Error	SS_E (by subtraction)	$(p - 3)(p - 1)$
Total	$SS_T = \sum_i \sum_j \sum_k \sum_l y_{ijkl}^2 - \frac{y_{....}^2}{N}$	$p^2 - 1$

Table 4.19

■ **TABLE 4.21**

Analysis of Variance for the Rocket Propellant Problem

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-Value
Formulations	330.00	4	82.50	10.00	0.0033
Batches of raw material	68.00	4	17.00		
Operators	150.00	4	37.50		
Test assemblies	62.00	4	15.50		
Error	66.00	8	8.25		
Total	676.00	24			

Table 4.21

Other Aspects of the RCBD

- The RCBD utilizes an additive model – no interaction between treatments and blocks
- Factorial design in Chapter 5 through 9
- Treatments and/or blocks as random effects
- Missing values
- Sample sizing in the RCBD? The OC curve approach can be used to determine the number of blocks to run

Random Blocks and/or Treatments

Assuming that the RCBD model Equation 4.1 is appropriate, if the blocks are random and the treatments are fixed we can show that:

$$\begin{aligned} E(y_{ij}) &= \mu + \tau_i, \qquad i = 1, 2, \ldots, a \\ V(y_{ij}) &= \sigma_\beta^2 + \sigma^2 \\ Cov(y_{ij}, y_{i'j'}) &= 0, \quad j \neq j' \\ Cov(y_{ij}, y_{i'j}) &= \sigma_\beta^2 \quad i \neq i' \end{aligned} \tag{4.14}$$

Thus, the variance of the observations is constant, the covariance between any two observations in different blocks is zero, but the covariance between two observations from the same block is σ_β^2. The expected mean squares from the usual ANOVA partitioning of the total sum of squares are

$$\begin{aligned} E(MS_{\text{Treatments}}) &= \sigma^2 + \frac{b\sum_{i=1}^{a}\tau_i^2}{a-1} \\ E(MS_{\text{Blocks}}) &= \sigma^2 + a\sigma_\beta^2 \\ E(MS_E) &= \sigma^2 \end{aligned} \tag{4.15}$$

The appropriate statistic for testing the null hypothesis of no treatment effects (all $\tau_i = 0$) is

$$F_0 = \frac{MS_{\text{Treatment}}}{MS_E}$$

which is exactly the same test statistic we used in the case where the blocks were fixed. Based on the expected mean squares, we can obtain an ANOVA-type estimator of the variance component for blocks as

$$\hat{\sigma}_\beta^2 = \frac{MS_{\text{Blocks}} - MS_E}{a} \tag{4.16}$$

For example, for the vascular graft experiment in Example 4.1 the estimate of σ_β^2 is

$$\hat{\sigma}_\beta^2 = \frac{MS_{\text{Blocks}} - MS_E}{a} = \frac{38.45 - 7.33}{4} = 7.78$$

TABLE 4.6
JMP Output for Example 4.1 with Blocks Assumed Random

Response Y

Summary of Fit

RSquare	0.756688
RSquare Adj	0.720192
Root Mean Square Error	2.706612
Mean of Response	89.79583
Observations (or Sum Wgts)	24

REML Variance Component Estimates

Random Effect	Var Ratio	Var Component	Std Error	95% Lower	95% Upper	Pct of Total
Block	1.0621666	7.7811667	6.116215	−4.206394	19.768728	51.507
Residual		7.32575	2.6749857	3.9975509	17.547721	48.493
Total		15.106917				100.000

Covariance Matrix of Variance Component Estimates

Random Effect	Block	Residual
Block	37.408085	−1.788887
Residual	−1.788887	7.1555484

Fixed Effect Tests

Source	Nparm	DF	DFDen	F Ratio	Prob > F
Pressure	3	3	15	8.1071	0.0019*

Choice of Sample Size (the # of blocks in RCBD)

Choice of Sample Size. Choosing the **sample size**, or the **number of blocks** to run, is an important decision when using an RCBD. Increasing the number of blocks increases the number of replicates and the number of error degrees of freedom, making design more sensitive. Any of the techniques discussed in Section 3.7 for selecting the number of replicates to run in a completely randomized single-factor experiment may be applied directly to the RCBD. For the case of a fixed factor, the operating characteristic curves in Appendix Chart V may be used with

$$\Phi^2 = \frac{b\sum_{i=1}^{a} \tau_i^2}{a\sigma^2} \qquad \textbf{(4.19)}$$

where there are $a - 1$ numerator degrees of freedom and $(a - 1)(b - 1)$ denominator degrees of freedom.

EXAMPLE 4.2

Consider the RCBD for the vascular grafts described in Example 4.1. Suppose that we wish to determine the appropriate number of blocks to run if we are interested in detecting a true maximum difference in yield of 6 with a reasonably high probability and an estimate of the standard deviation of the errors is $\sigma = 3$. From Equation 3.45, the minimum value of Φ^2 is (writing b, the number of blocks, for n)

$$\Phi^2 = \frac{bD^2}{2a\sigma^2}$$

where D is the maximum difference we wish to detect. Thus,

$$\Phi^2 = \frac{b(6)^2}{2(4)(3)^2} = 0.5b$$

If we use $b = 5$ blocks, $\Phi = \sqrt{0.5b} = \sqrt{0.5(5)} = 1.58$, and there are $(a-1)(b-1) = 3(4) = 12$ error degrees of freedom. Appendix Chart V with $\nu_1 = a - 1 = 3$ and $\alpha = 0.05$ indicates that the β risk for this design is approximately 0.55 (power $= 1 - \beta = 0.45$). If we use $b = 6$ blocks, $\Phi = \sqrt{0.5b} = \sqrt{0.5(6)} = 1.73$, with $(a-1)(b-1) = 3(5) = 15$ error degrees of freedom, and the corresponding β risk is approximately 0.4 (power $= 1 - \beta = 0.6$). Because the batches of resin are expensive and the cost of experimentation is high, the experimenter decides to use six blocks, even though the power is only about 0.6 (actually many experiments work very well with power values of only 0.5 or higher).

Balanced Incomplete Block Designs

- Sometimes, it is not practical to run all treatment combinations in each block
- Randomized incomplete block designs: cannot fit all treatments in each block
- Balanced incomplete block design (BIBD): is an incomplete block design in which any two treatments appear together an equal number of times

Identify a, b, k, r, and λ for the following examples.

Example 1:

	Block	
1	2	3
A	B	A
B	C	C

Example 2:

		Block			
1	2	3	4	5	6
A	A	A	B	B	C
B	C	D	C	D	D

Example 3:

		Block	
1	2	3	4
A	A	A	B
B	B	C	C
C	D	D	D

Example: BIBD

Suppose that a chemical engineer thinks that the time of reaction for a chemical process is a function of the type of catalyst employed. Four catalysts are currently being investigated. The experimental procedure consists of selecting a batch of raw material, loading the pilot plant, applying each catalyst in a separate run of the pilot plant, and observing the reaction time. Because variations in the batches or raw material may affect the performance of the catalysts, the engineer decides to use batches of raw material as blocks. However, each batch is only large enough to permit three catalysts to be run. The order in which the catalysts are run in each block is randomized.

TABLE 4.22
Balanced Incomplete Block Design for Catalyst Experiment

Treatment (Catalyst)	Block (Batch of Raw Material) 1	2	3	4	$y_{i.}$
1	73	74	—	71	218
2	—	75	67	72	214
3	73	75	68	—	216
4	75	—	72	75	222
$y_{.j}$	221	224	207	218	$870 = y_{..}$

ANOVA Table for Balanced Incomplete Block Design

■ **TABLE 4.23**

Analysis of Variance for the Balanced Incomplete Block Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments (adjusted)	$\frac{k \sum Q_i^2}{\lambda a}$	$a - 1$	$\frac{SS_{\text{Treatments(adjusted)}}}{a - 1}$	$F_0 = \frac{MS_{\text{Treatments(adjusted)}}}{MS_E}$
Blocks	$\frac{1}{k} \sum y_{.j}^2 - \frac{y_{..}^2}{N}$	$b - 1$	$\frac{SS_{\text{Blocks}}}{b - 1}$	
Error	SS_E (by subtraction)	$N - a - b + 1$	$\frac{SS_E}{N - a - b + 1}$	
Total	$\sum \sum y_{ij}^2 - \frac{y_{..}^2}{N}$	$N - 1$		

Table 4.23

EXAMPLE 4.5

Consider the data in Table 4.22 for the catalyst experiment. This is a BIBD with $a = 4$, $b = 4$, $k = 3$, $r = 3$, $\lambda = 2$, and $N = 12$. The analysis of this data is as follows. The total sum of squares is

$$SS_T = \sum_i \sum_j y_{ij}^2 - \frac{y_{..}^2}{12}$$

$$= 63{,}156 - \frac{(870)^2}{12} = 81.00$$

The block sum of squares is found from Equation 4.33 as

$$SS_{\text{Blocks}} = \frac{1}{3}\sum_{j=1}^{4} y_{.j}^2 - \frac{y_{..}^2}{12}$$

$$= \frac{1}{3}[(221)^2 + (207)^2 + (224)^2 + (218)^2] - \frac{(870)^2}{12}$$

$$= 55.00$$

To compute the treatment sum of squares adjusted for blocks, we first determine the adjusted treatment totals using Equation 4.35 as

$$Q_1 = (218) - \tfrac{1}{3}(221 + 224 + 218) = -9/3$$

$$Q_2 = (214) - \tfrac{1}{3}(207 + 224 + 218) = -7/3$$

$$Q_3 = (216) - \tfrac{1}{3}(221 + 207 + 224) = -4/3$$

$$Q_4 = (222) - \tfrac{1}{3}(221 + 207 + 218) = 20/3$$

The adjusted sum of squares for treatments is computed from Equation 4.34 as

$$SS_{\text{Treatments(adjusted)}} = \frac{k\sum_{i=1}^{4} Q_i^2}{\lambda a}$$

$$= \frac{3[(-9/3)^2 + (-7/3)^2 + (-4/3)^2 + (20/3)^2]}{(2)(4)}$$

$$= 22.75$$

The error sum of squares is obtained by subtraction as

$$SS_E = SS_T - SS_{\text{Treatments(adjusted)}} - SS_{\text{Blocks}}$$

$$= 81.00 - 22.75 - 55.00 = 3.25$$

The analysis of variance is shown in Table 4.24. Because the P-value is small, we conclude that the catalyst employed has a significant effect on the time of reaction.

■ TABLE 4.24

Analysis of Variance for Example 4.5

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-Value
Treatments (adjusted for blocks)	22.75	3	7.58	11.66	0.0107
Blocks	55.00	3	—		
Error	3.25	5	0.65		
Total	81.00	11			